

Rational Spline with Interval and Point Tension

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Abstract

Various curve designing methods have been developed, for the designing of distinct objects, for applications like font designing, Computer Aided Design (CAD), Computer Aided Engineering (CAE), etc. Some methods are better suited for controlling the shape of the curve on an interval, while others are better suited for controlling the shape at the individual control points. In this paper, a rational cubic C^2 spline curve is described which has interval and point tension weights for manipulating the shape of the curve. The spline is presented in both interpolatory and local support basis form, and the effect of the weights on these representations is analyzed.

Keywords

Rational cubic spline, rational Bernstein – Bézier curves, non – uniform rational B – Spline, shape control, tension.

1. Introduction

Interpolation techniques play a very important role in obtaining solutions of various problems that arise in many areas of scientific computation. Generally interpolant is preferred which preserves some of the characteristics of the function to be interpolated. In order to tackle such situations, a variety of shape preserving interpolation methods have been discussed in the literature for interpolating given sets of monotonic data.

In this paper we have discussed parametric C^2 rational cubic spline representation having interval and point tension weights. These weights help us in playing with the shape of the curve without disturbing the point data. In earlier research the splines were developed with shape control parameter using the geometric GC^2 continuity constraints, which could be extended to C^2 continuity with respect to re-parameterization as discussed by [Neil74], [Bars81] and [Boeh85]. [Dege88]

contribution leads towards the fact that C^2 rational cubic splines can be used as an alternative to the use of geometric GC^2 cubic or GC^2 rational cubic spline representations. Moreover the rational splines also provide a C^2 alternative to the C^1 weighted v -spline of [Fole87] and [Fole88].

In section 2 we discuss the rational cubic form of the splines, which is the basis of rest of the technique. Section 3 is about the rational cubic spline interpolant form. Local support is discussed in section 4. In section 5 we have presented the results. We close the paper with conclusion and some future directions in section 6.

2. The Rational Cubic Form

Let $\mathbf{F}_i \in \mathbf{R}^m$, $i \in \mathbf{Z}$, be values given at the distinct knots $t_i \in \mathbf{R}$, $i \in \mathbf{Z}$, with interval spacing $h_i := t_{i+1} - t_i > 0$. Also let $\mathbf{D}_i \in \mathbf{R}^m$, $i \in \mathbf{Z}$, denote first derivative values defined at the knots. Then a parametric C^1 piecewise rational cubic Hermite function $p : \mathbf{R} \rightarrow \mathbf{R}^m$ is defined by

$$p|_{[t_i, t_{i+1})}(t) = \frac{[(1-\theta)^3 \alpha_i F_i + \theta(1-\theta)^2 (\alpha_i + \gamma_i) V_i + \theta^2 (1-\theta) (\beta_i + \gamma_i) W_i + \theta^3 \beta_i F_{i+1}]}{(1-\theta)^2 \alpha_i + \gamma_i \theta (1-\theta) + \theta^2 \beta_i} \quad (2.1)$$

where
$$\theta_{[t_i, t_{i+1})}(t) = \frac{(t - t_i)}{h_i} \quad (2.2)$$

and
$$V_i = F_i + \frac{\alpha_i}{\alpha_i + \gamma_i} h_i D_i, \quad W_i = F_{i+1} - \frac{\beta_i}{\beta_i + \gamma_i} h_i D_{i+1} \quad (2.3)$$

We have made use of rational Bernstein – Bézier representation, where the control points $\{F_i, V_i, W_i, F_{i+1}\}$ are determined by imposing the Hermite interpolation conditions

$$p(t_i) = \mathbf{F}_i \text{ and } p^{(1)}(t_i) = \mathbf{D}_i, i \in \mathbf{Z}.$$

The scalar weights in the numerator of (2.1) are those given by degree raising the denominator to cubic form, since

$$(1-\theta)^2 \alpha_i + \gamma_i \theta (1-\theta) + \theta^2 \beta_i = (1-\theta)^3 \alpha_i + \theta(1-\theta)^2 (\alpha_i + \gamma_i) + \theta^2 (1-\theta) (\beta_i + \gamma_i) + \theta^3 \beta_i \quad (2.4)$$

Since the denominator is positive, it follows from Bernstein – Bézier theory that the curve segment $p|_{[t_i, t_{i+1})}$ lies in the convex hull of the control points $\{F_i, V_i, W_i, F_{i+1}\}$ and is variation diminishing with respect to the ‘control polygon’ joining these points.

For practical implementation we will take $\alpha_i = 1/\lambda_i$ and $\beta_i = 1/\mu_i$. (2.5)

This leads to a consistent behavior with respect to increasing weights and avoids numerical problems associated with evaluation at $\theta = 0$ and $\theta = 1$ in the (removable) singular cases $\alpha_i = 0$, $\beta_i = 0$. We now have,

$$V_i = F_i + \frac{1}{\lambda_i \gamma_i + 1} h_i D_i, \quad W_i = F_{i+1} - \frac{1}{\mu_i \gamma_i + 1} h_i D_{i+1} \quad (2.6)$$

The following ‘tension’ properties of the rational Hermite form are now immediately apparent from (2.1), (2.5), and (2.6):

1. Point tension

$$\lim_{\lambda_i \rightarrow \infty} V_i = F_i \text{ and}$$

$$\lim_{\lambda_i \rightarrow \infty} p|_{[t_i, t_{i+1})}(t) = \frac{(1-\theta)^2 \gamma_i F_i + \theta(1-\theta)(\beta_i + \gamma_i)W_i + \theta^2 \beta_i F_{i+1}}{\gamma_i(1-\theta) + \theta \beta_i} \quad (2.7)$$

$$\lim_{\mu_i \rightarrow \infty} W_i = F_{i+1} \text{ and}$$

$$\lim_{\mu_i \rightarrow \infty} p|_{[t_i, t_{i+1})}(t) = \frac{(1-\theta)^2 \alpha_i F_i + \theta(1-\theta)(\alpha_i + \gamma_i)V_i + \theta^2 \gamma_i F_{i+1}}{(1-\theta)\alpha_i + \gamma_i \theta} \quad (2.8)$$

a. *Biased point tension*

$$\mu_i \rightarrow \infty \text{ or } \lambda_i \rightarrow \infty$$

b. *Accentuated point tension*

$$\mu_{i-1} = \lambda_i \rightarrow \infty$$

2. Interval tension

$$\lim_{\gamma_i \rightarrow \infty} V_i = F_i, \quad \lim_{\gamma_i \rightarrow \infty} W_i = F_{i+1} \text{ and}$$

$$\lim_{\gamma_i \rightarrow \infty} p|_{[t_i, t_{i+1})}(t) = (1-\theta)F_i + \theta F_{i+1} \quad (2.9)$$

Interval tension property can also be recovered by letting $\lambda_i, \mu_i \rightarrow \infty$.

3. Rational Cubic Spline Interpolant

We now consider the problem of constructing a parametric C^2 rational cubic spline interpolant on the interval $[t_0, t_n]$, using the rational cubic Hermite form of section 2. This is the situation where $\mathbf{F}_i \in \mathbf{R}^m$, $i = 0, \dots, n$, are the given interpolation data at knots t_i , $i = 0, \dots, n$, and the derivatives $\mathbf{D}_i \in \mathbf{R}^m$, $i = 0, \dots, n$, are degrees of freedom to be determined by the imposition of C^2 constraints on the piecewise defined rational Hermite form. \mathbf{D}_0 and \mathbf{D}_n are assumed to be given as end conditions.

The C^2 constraints

$$p^{(2)}(t_i^+) = p^{(2)}(t_i^-), \quad i = 0, \dots, n-1 \quad (3.1)$$

gives the tri – diagonal system of ‘consistency equations’

$$\frac{h_i \alpha_{i-1}}{\beta_{i-1}} \mathbf{D}_{i-1} + \left(\frac{h_i \gamma_{i-1}}{\beta_{i-1}} + \frac{h_{i-1} \gamma_i}{\alpha_i} \right) \mathbf{D}_i + \frac{h_{i-1} \beta_i}{\alpha_i} \mathbf{D}_{i+1} = \frac{h_i (\gamma_{i-1} + \alpha_{i-1})}{\beta_{i-1}} \Delta_{i-1} + \frac{h_{i-1} (\gamma_i + \beta_i)}{\alpha_i} \Delta_i, \quad (3.2)$$

$$i = 0, \dots, n-1$$

where

$$\Delta_i := (\mathbf{F}_{i+1} - \mathbf{F}_i) / h_i \quad (3.3)$$

Thus, in terms of the reciprocal weights (2.5), the tri – diagonal system is

$$\frac{h_i \mu_{i-1}}{\lambda_{i-1}} \mathbf{D}_{i-1} + (h_i \gamma_{i-1} \mu_{i-1} + h_{i-1} \gamma_i \lambda_i) \mathbf{D}_i + \frac{h_{i-1} \lambda_i}{\mu_i} \mathbf{D}_{i+1} = h_i (\gamma_{i-1} + 1/\lambda_{i-1}) \mu_{i-1} \Delta_{i-1} + h_{i-1} (\gamma_i + 1/\mu_i) \lambda_i \Delta_i, \quad (3.4)$$

$$i = 0, \dots, n-1$$

4. The Local Support Basis

We now seek a local support basis representation for the space of C^2 rational cubic splines.

Imposing the constraint

$$p^{(1)}(t_i^+) = p^{(1)}(t_i^-) \quad (4.1)$$

on the rational Bernstein – Bézier form in (2.1) gives

$$\mathbf{F}_i = (1 - \delta_i)\mathbf{W}_{i-1} + \delta_i\mathbf{V}_i \quad (4.2)$$

where

$$\delta_i := \frac{h_{i-1}(\gamma_i \lambda_i + 1)}{h_{i-1}(\gamma_i \lambda_i + 1) + h_i(\gamma_{i-1} \mu_{i-1} + 1)} \quad (4.3)$$

Also, imposing the constraints

$$\mathbf{p}^{(2)}(\mathbf{t}_i^+) = \mathbf{p}^{(2)}(\mathbf{t}_i^-) \quad (4.4)$$

gives

$$a_{2,i}\mathbf{F}_i - (a_{2,i} + a_{1,i})\mathbf{V}_i + a_{1,i}\mathbf{W}_i = b_{2,i-1}\mathbf{F}_i - (b_{2,i-1} + b_{1,i-1})\mathbf{W}_{i-1} + b_{1,i-1}\mathbf{V}_{i-1} \quad (4.5)$$

where

$$a_{1,i} = \frac{2}{h_i^2}(\gamma_i \lambda_i + \lambda_i / \mu_i), \quad a_{2,i} = \frac{2}{h_i^2}(\gamma_i^2 \lambda_i^2 + \lambda_i / \mu_i) \quad (4.6)$$

$$b_{1,i} = \frac{2}{h_i^2}(\gamma_i \mu_i + \mu_i / \lambda_i), \quad b_{2,i} = \frac{2}{h_i^2}(\gamma_i^2 \mu_i^2 + \mu_i / \lambda_i) \quad (4.7)$$

Eliminating \mathbf{F}_i using (4.2) then gives

$$(1 - \tau_{i-1})\mathbf{W}_{i-1} + \tau_{i-1}\mathbf{V}_{i-1} = (1 - \sigma_i)\mathbf{V}_i + \sigma_i\mathbf{W}_i \quad (4.8)$$

where

$$\sigma_i := \frac{-a_{1,i}}{a_{2,i}(1 - \delta_i) + b_{2,i-1}\delta_i} \quad \tau_i := \frac{-b_{1,i}}{a_{2,i+1}(1 - \delta_{i+1}) + b_{2,i}\delta_{i+1}} \quad (4.9)$$

Equation (4.8) represents the equation for the intersection of the two lines through \mathbf{W}_{i-1} , \mathbf{V}_{i-1} and \mathbf{V}_i , \mathbf{W}_i respectively; actually this intersection is the control point \mathbf{c}_i of the local support basis representation.

Thus, given $\{\mathbf{c}_i\}_{i \in \mathbf{Z}}$ we have

$$\left. \begin{aligned} (1 - \sigma_i)\mathbf{V}_i + \sigma_i\mathbf{W}_i &= \mathbf{c}_i, \\ \tau_i\mathbf{V}_i + (1 - \tau_i)\mathbf{W}_i &= \mathbf{c}_{i+1}, \end{aligned} \right\} \quad i \in \mathbf{Z}, \quad (4.10)$$

and hence

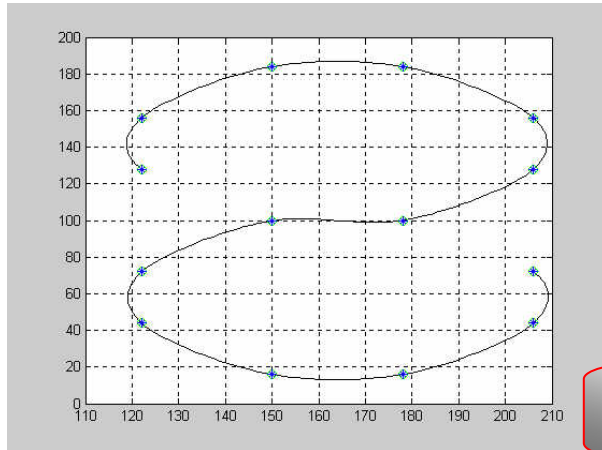
$$\left. \begin{aligned} V_i &= [(1-\tau_i)/\Delta_i]c_i - [\sigma_i/\Delta_i]c_{i+1}, \\ W_i &= -[\tau_i/\Delta_i]c_i - [(1-\sigma_i)/\Delta_i]c_{i+1}, \end{aligned} \right\} \quad i \in \mathbf{Z}, \quad (4.11)$$

where

$$\Delta_i := 1 - \sigma_i - \tau_i$$

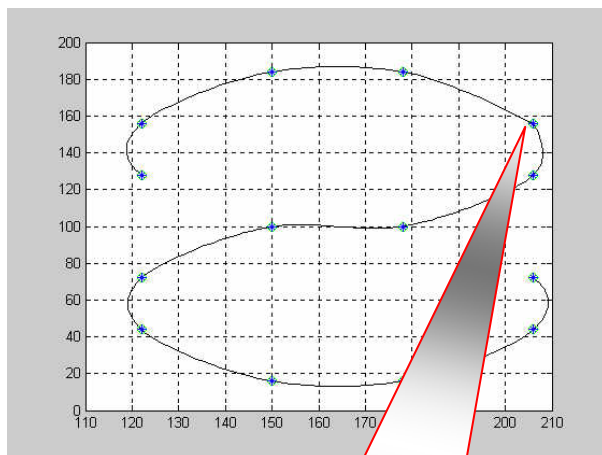
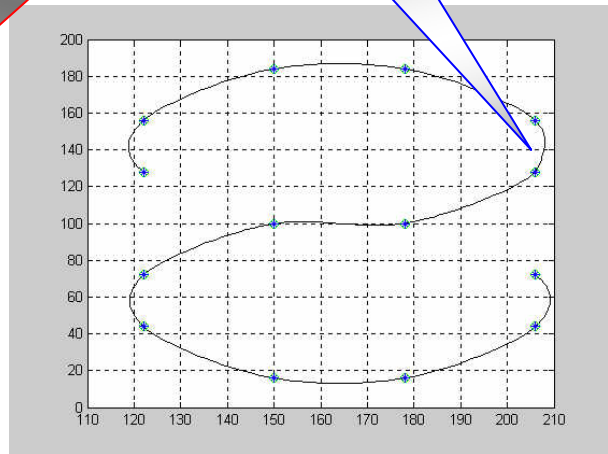
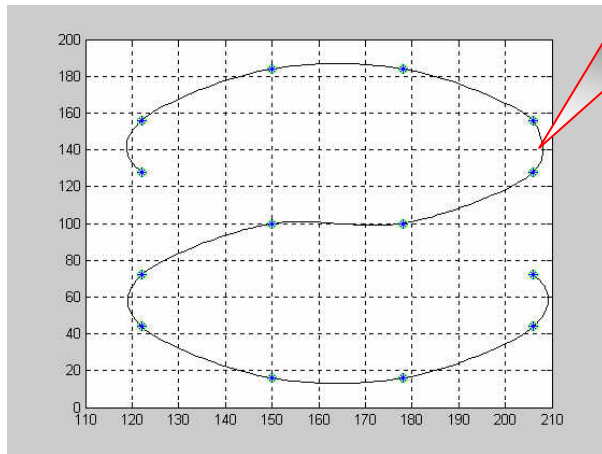
5. Simulated Results

1. Rational Cubic C^1 Hermit form:

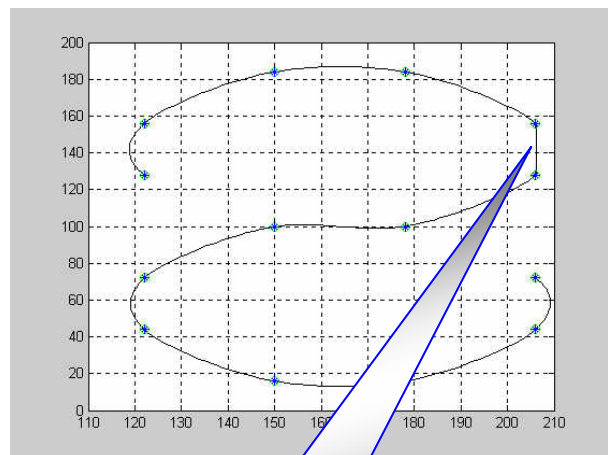


Biased Point Tension
 $\mu_5 = 100$

Biased Point Tension
 $\lambda_5 = 100$

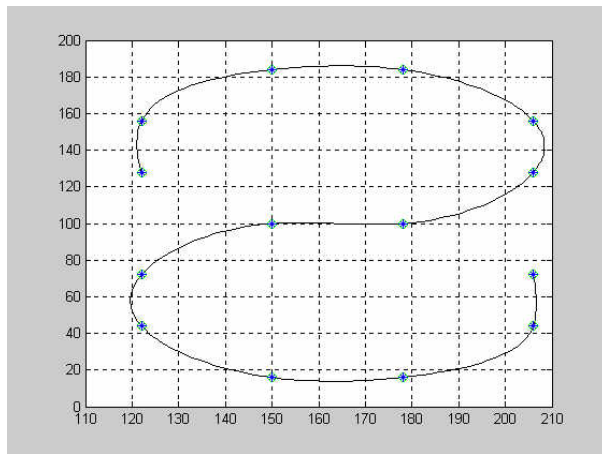


Accentuated Point Tension
 $\mu_4 = \lambda_5 = 100$



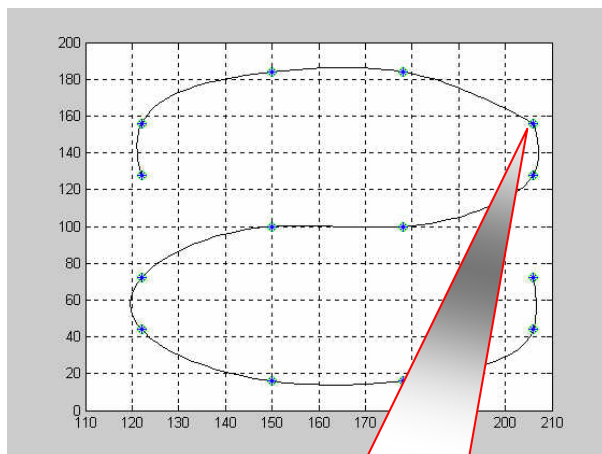
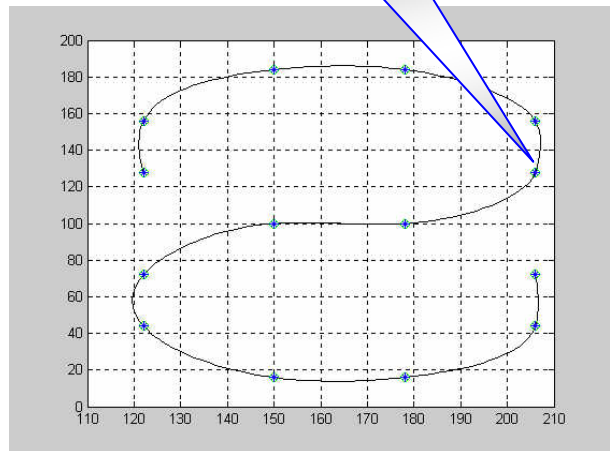
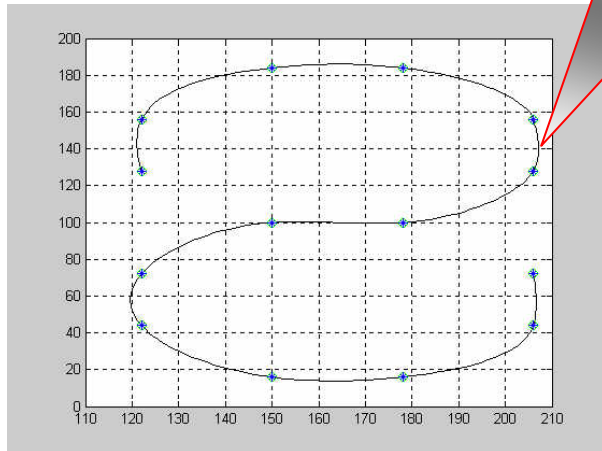
Interval Tension
 $\gamma_5 = 100$

2. Rational Cubic C^2 Interpolant form:

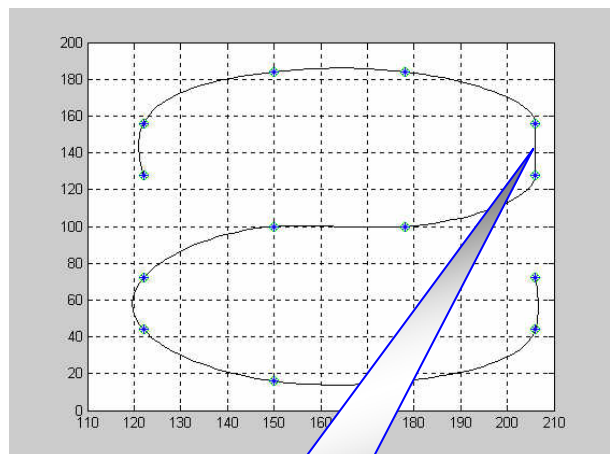


Biased Point Tension
 $\mu_5 = 100$

Biased Point Tension
 $\lambda_5 = 100$

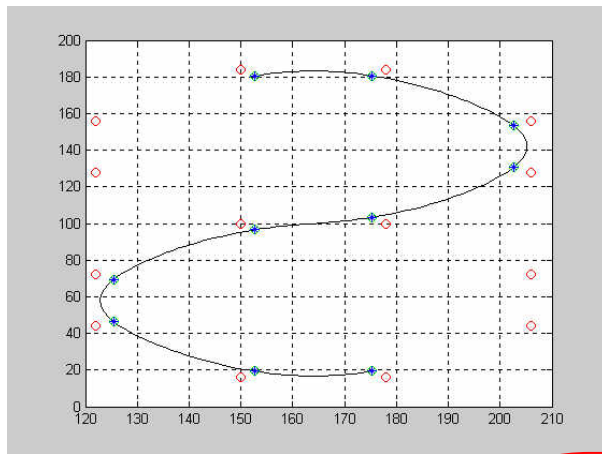


Accentuated Point Tension
 $\mu_4 = \lambda_5 = 100$

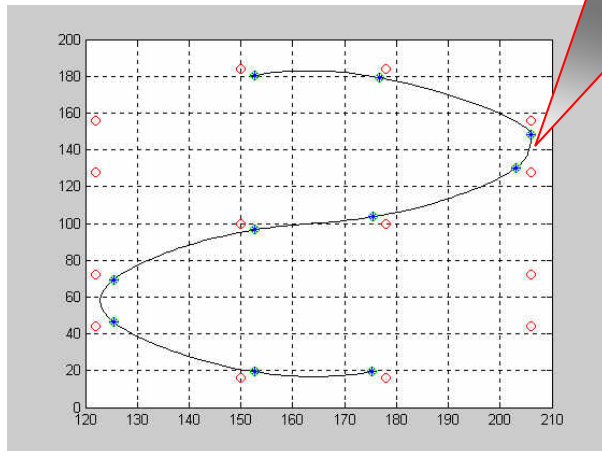


Interval Tension
 $\gamma_5 = 100$

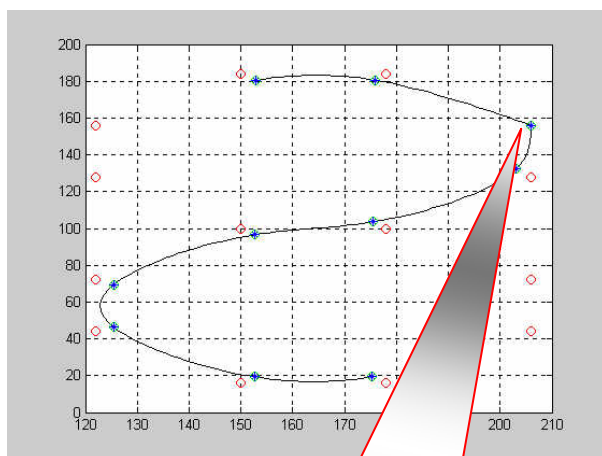
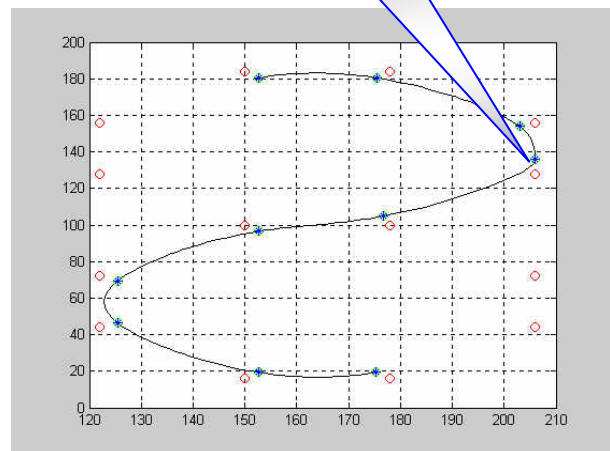
3. Local Support Basis form with Point and Interval Tension:



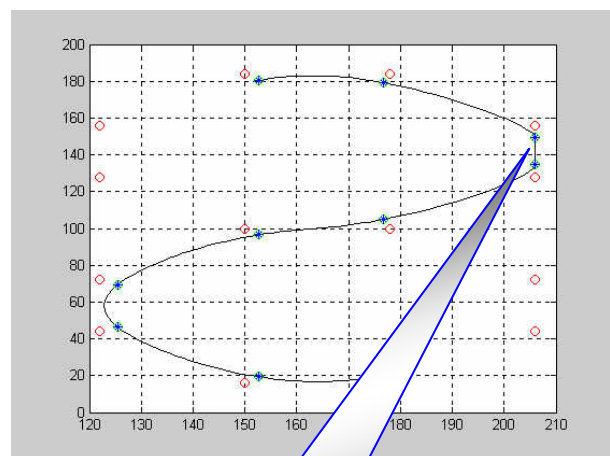
Biased Point Tension
 $\lambda_5 = 100$



Biased Point Tension
 $\mu_5 = 100$

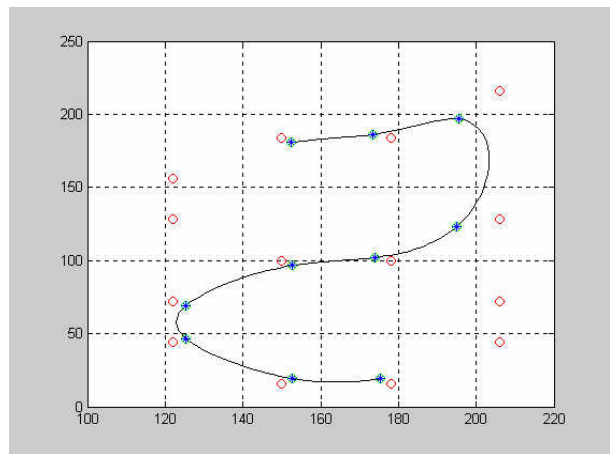
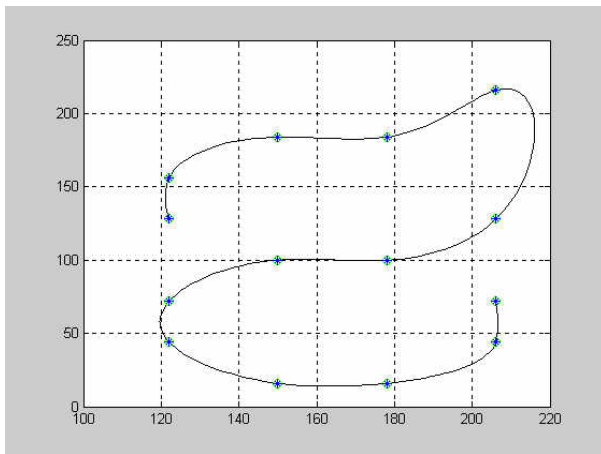
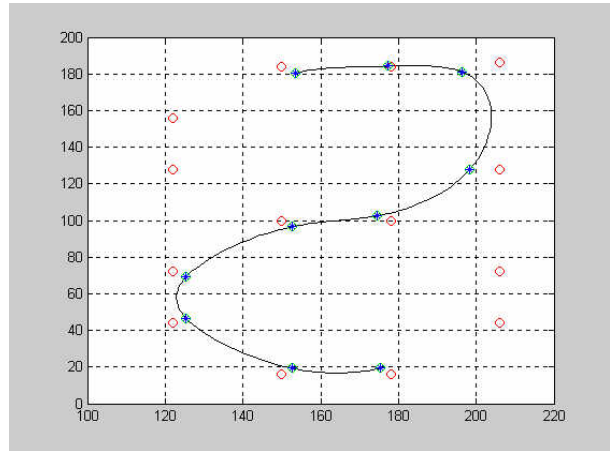
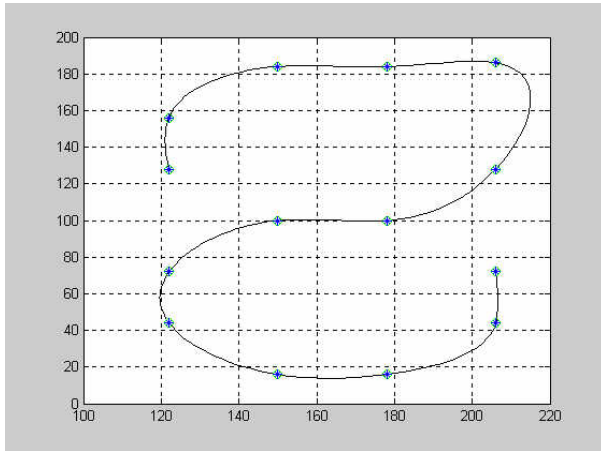


Accentuated Point Tension
 $\mu_4 = \lambda_5 = 100$



Interval Tension
 $\gamma_5 = 100$

3. Local Support Basis form with Data Point Placement:



6. Conclusion

We have described and shown the results of C^2 rational cubic spline curve representation, which has interval and point tension weights. We have also demonstrated in this paper the affects of playing with the shape parameters without changing the data set. This shows the control achieved on the curve on the whole through segmentation. We have also presented the interpolant and local support form of the curve. This work can be extended to generate surfaces with controlled behavior. One possible solution to achieve the goal of point and interval tension property for the surfaces could be to construct the Boolean sun, spline-blended, rectangular network of parametric v -spline curves as suggested by [Neil86]. Another approach could be to use polygonal faces rather than using rectangular faces for tensor product or Boolean sum. This technique is given by [Greg92].

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